Stochastic Processes and their Applications 25 (1987) 203-213
North-Holland

A NOTE ON THE STABILITY OF THE LOCAL TIME OF A WIENER PROCESS

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Received 6 November 1985
Revised 18 December 1986

Let $L(a, t)$ be the local time of a Wiener process, and put

$$A(t)=\sup_{|a|\leqslant g(t)}\left|\frac{L(a,t)}{L(0,t)}-1\right|.$$

It is shown that if $g(t)=t^{1/2}(\log t)^{-1}(\log\log t)^{-\rho}$

$$\lim_{t\to\infty} A(t)=0 \text{ a.s. when } \rho>2$$

and

$$\limsup_{t\to\infty} A(t)\geqslant 1 \text{ a.s. when } \rho=1.$$

A similar result is proved for random $g(t)$ depending on the maximum of the Wiener process. These results settle a problem posed by Csörgő and Révész [7].

AMS 1980 Subject Classifications: Primary 60J55; Secondary 60J65, 60J60.

local time * Wiener process * diffusion

Introduction

Let $X_1, X_2, \ldots$ be i.i.d. random variables with $P(X_1=+1)=P(X_1=-1)=\frac{1}{2}$. Put $S_0=0$, $S_n=X_1+X_2+\cdots+X_n$ $(n\geqslant 1)$. Define the local time of the symmetric random walk $\{S_k\}$ by

$$\xi(x,n)=\#\{k\colon 0\leqslant k\leqslant n,\ S_k=x\}$$

$(n=1,2,\ldots,\ x=0,\pm1,\pm2,\ldots)$.

Let $\{W(t),\ t\geqslant 0\}$ be a Wiener process, and let

$$H(A,t)=\lambda\{s;\ 0\leqslant s\leqslant t,\ W(s)\in A\}$$

be the occupation time of $W(\cdot)$, where λ is the Lebesque measure. Its Radon-Nikodym derivate $L(a,t)$ defined by

$$H(A,t)=\int_A L(a,t)\,\mathrm{d}a$$

is called the local time of $W(t)$. It is well known that $L(a, t)$ exists and is jointly continuous in (a, t) with probability 1.

In their paper Csörgő and Révész [7] proved the following two theorems.

Theorem A. *For any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \sup_{|k|\le a_n} \left| \frac{\xi(k, n)}{\xi(0, n)} - 1 \right| = 0 \quad a.s.$$

where $a_n = n^{1/2}(\log n)^{-(2+\varepsilon)}$.

Theorem B

$$\limsup_{n\to\infty} \sup_{|k|\le b_n} \left| \frac{\xi(k, n)}{\xi(0, n)} - 1 \right| \ge 1 \quad a.s.$$

where $b_n = n^{1/2}(\log n)^{-1}$.

Moreover they presented the following conjecture.

Conjecture

$$\lim_{n\to\infty} \sup_{m_n/\log\log n \le k \le M_n/\log\log n} \left| \frac{\xi(k, n)}{\xi(0, n)} - 1 \right| = 0 \quad a.s.,$$

where $m_n = \min_{0\le k\le n} S_k$, $M_n = \max_{0\le k\le n} S_k$.

In this note our aim is to narrow the gap between Theorems A and B and show that, though the conjecture is not true, a modified form of it holds, namely the factor $(\log\log n)$ should be replaced by $(\log n)(\log\log n)^{5/2+\varepsilon}$.

We formulate our results for $L(a, t)$, but it can be seen from the following invariance principle due to Révész [13] that similar results hold for $\xi(x, n)$ as well.

Theorem C. *On a probability space* $(\Omega, \mathcal{A}, P)$ *one can define a Wiener process and a sequence* $X_1, X_2, \ldots$ *of i.i.d. random variables with* $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$, *such that for any* $\delta > 0$,

$$\lim_{n\to\infty} n^{-1/4-\delta} \sup_x |\xi(x, n) - L(x, n)| = 0 \quad a.s.$$

and

$$\lim_{n\to\infty} n^{-1/4-\delta} |S_n - W(n)| = 0 \quad a.s.$$

simultaneously.

By using more general invariance principles of similar type (simultaneously for local time and partial sums) one easily sees that our results can be reformulated for more general case (cf. Csáki and Révész [6], Csörgő and Révész [8], Borodin [3]).

1. Main results

Theorem 1. *Put*

$$g(t)=\frac{\sqrt{t}}{(\log t)(\log\log t)^{\rho}}.$$

Then

$$\text{(i)}\quad \lim_{t\to\infty}\sup_{|a|\leq g(t)}\left|\frac{L(a,t)}{L(0,t)}-1\right|=0 \quad \textit{a.s. if } \rho>2,$$

$$\text{(ii)}\quad \limsup_{t\to\infty}\sup_{|a|\leq g(t)}\left|\frac{L(a,t)}{L(0,t)}-1\right|\geq 1 \quad \textit{a.s. if } \rho=1.$$

Theorem 2. *Put*

$$h_1(t)=\frac{m(t)}{(\log t)(\log\log t)^{\rho}}$$

and

$$h_2(t)=\frac{M(t)}{(\log t)(\log\log t)^{\rho}},$$

where

$$m(t)=\inf_{0\leq s\leq t} W(s),\qquad M(t)=\sup_{0\leq s\leq t} W(s).$$

Then

$$\text{(i)}\quad \lim_{t\to\infty}\sup_{h_1(t)\leq a\leq h_2(t)}\left|\frac{L(a,t)}{L(0,t)}-1\right|=0 \quad \textit{a.s. if } \rho>\tfrac{5}{2},$$

$$\text{(ii)}\quad \limsup_{t\to\infty}\sup_{ch_1(t)\leq a\leq ch_2(t)}\left|\frac{L(a,t)}{L(0,t)}-1\right|=\infty \quad \textit{a.s. if } \rho=0$$

and $c>0$ is any constant.

It is obvious that (i) of Theorem 1 implies (i) of Theorem 2 by the law of the iterated logarithm for the Wiener process. On the other hand, (ii) of Theorem 1 can be proved in the same way as Theorem B has been proved in Csörgő and Révész [7].

Thus only (i) of Theorem 1 and (ii) of Theorem 2 will be proved.

2. Proof of Theorem 1(i)

Our proof is based on a result of Bass and Griffin [1]. They proved (Lemma 3.4) that if

$$T_r=\inf\{t\colon L(0,t)\geq r\}$$

then there exists a constant c such that for all h and λ sufficiently small,

$$P\left(\sup_{1\leq r\leq 2}\sup_{0\leq a\leq h}(L(a,T_r)-r)>\lambda\right)\leq c\frac{\lambda}{h}\exp\left(-\frac{\lambda^2}{32h}\right). \tag{2.1}$$

By using this result we prove

Lemma 2.1. *For any $\beta>1$,*

$$\lim_{r\to\infty}\sup_{0\leq a\leq r/(\log\log r)^\beta}\frac{L(a,T_r)-r}{r}=0 \quad a.s. \tag{2.2}$$

Proof. Following the proof of Proposition 3.5 in Bass and Griffin [1], we get from (2.1)

$$\begin{aligned}
&P\left(\sup_{2^n\leq r<2^{n+1}}\sup_{0\leq a\leq r/(\log\log r)^\beta}\left(\frac{L(a,T_r)-r}{r}\right)>\varepsilon\right)\\
&=P\left(\sup_{1\leq r<2}\sup_{0\leq a\leq r/(\log\log(2^n r))^\beta}\left(\frac{L(a,T_r)-r}{r}\right)>\varepsilon\right)\\
&\leq P\left(\sup_{1\leq r<2}\sup_{0\leq a\leq 1/(\log\log 2^n)^\beta}\left(\frac{L(a,T_r)-r}{r}\right)>\varepsilon\right)\\
&\leq P\left(\sup_{1\leq r<2}\sup_{0\leq a\leq 2/(\log\log 2^n)^\beta}(L(a,T_r)-r)>\varepsilon\right)\\
&\leq c\varepsilon(\log\log 2^n)^\beta\exp\left\{-\frac{\varepsilon^2}{64}(\log\log 2^n)^\beta\right\}.
\end{aligned}$$

Since this is a term of a convergent series if $\beta>1$, (2.2) follows by Borel-Cantelli lemma.

Hence we obtain that for any $\varepsilon>0$ and r large enough,

$$\sup_{0\leq a\leq r/(\log\log r)^\beta}L(a,T_r)\leq(1+\varepsilon)r \quad \text{a.s.} \tag{2.3}$$

Put $r=r_t=L(0,t)+1$. Then $T_{r_t}\geq t$ and

$$r_t/(\log\log r_t)^\beta\geq\frac{L(0,t)}{(\log\log L(0,t))^\beta} \quad \text{a.s.}$$

if t is large enough.

Moreover, from the well known integral test of Chung and Hunt [4] for any $\alpha>1$,

$$L(0,t)\geq\frac{\sqrt{t}}{(\log t)(\log\log t)^\alpha} \quad \text{a.s.}$$

if t is big enough. Hence

$$\sup_{0\leq a\leq r_t/(\log\log r_t)^\beta}L(a,T_{r_t})\geq\sup_{0\leq a\leq\sqrt{(t)}/(\log t)(\log\log t)^{\alpha+\beta}}L(a,t) \quad \text{a.s.}$$

if t is big enough. Thus from (2.3) we have

$$\sup_{0\leq a\leq g(t)} L(a,t)\leq(1+\varepsilon)(L(0,t)+1) \quad \text{a.s.}$$

with $g(t)$ defined in Theorem 1, $\rho=\alpha+\beta$. Since $\varepsilon>0$ is arbitrary, we have

$$\lim_{t\to\infty}\sup_{0\leq a\leq g(t)}\left(\frac{L(a,t)}{L(0,t)}-1\right)=0 \quad \text{a.s.}$$

From the symmetry of the Wiener process it follows immediately that

$$\lim_{t\to\infty}\sup_{|a|\leq g(t)}\left(\frac{L(a,t)}{L(0,t)}-1\right)=0 \quad \text{a.s.}$$

A similar argument shows

$$\lim_{t\to\infty}\sup_{|a|\leq g(t)}\left(1-\frac{L(a,t)}{L(0,t)}\right)=0 \quad \text{a.s.}$$

This completes the proof of Theorem 1(i).

3. Proof of Theorem 2(ii)

It is a well known result of Ray [12], that if T is an exponential random variable with parameter $\lambda>0$, and independent of $W(t)$ then $L(a,T)$ under the condition $W(T)=z>0$ is a diffusion in a, with generator (see e.g. Borodin [2])

$$2x\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\sqrt{2\lambda}\,\frac{\mathrm{d}}{\mathrm{d}x}\right) \quad \text{if } x>z>0 \text{ or } x<0 \tag{3.1}$$

and with generator

$$2x\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\sqrt{2\lambda}\,\frac{\mathrm{d}}{\mathrm{d}x}\right)+2\frac{\mathrm{d}}{\mathrm{d}x} \quad \text{if } 0<x<z. \tag{3.2}$$

Moreover, with the notation $\beta=\sqrt{2\lambda}$,

$$P(L(0,T)\in[x,x+\mathrm{d}x)\mid W(T)=z)=\beta\,\mathrm{e}^{-\beta x}\,\mathrm{d}x. \tag{3.3}$$

Hence, for the Laplace transform of the first passage time

$$\tau_s=\inf\{x;\ L(x,T)=s\},$$

we have (see Itô and McKean [9, p. 145])

$$E_a(\mathrm{e}^{-\alpha\tau_b})=\frac{g(a)}{g(b)} \quad \text{for } 0<a<b \tag{3.4}$$

where $g(\cdot)$ is the strictly increasing solution of

$$2x\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}y-\sqrt{2\lambda}\,\frac{\mathrm{d}}{\mathrm{d}x}y\right)+2\frac{\mathrm{d}}{\mathrm{d}x}y=\alpha y. \tag{3.5}$$

From Kamke [10, p. 473-475]

$$g(x) = {}_1F_1\left(\frac{\alpha}{2\beta}, 1, \beta x\right)$$

where ${}_1F_1(r, s, x)$ is the confluent hypergeometric function

$${}_1F_1(r, s, x) = 1 + \frac{r}{s}x + \frac{(r)_2}{(s)_2}\frac{x^2}{2!} + \cdots + \frac{(r)_n}{(s)_n}\frac{x^n}{n!} + \cdots$$

and $(r)_n = r(r+1)(r+2)\cdots(r+n-1)$. Consequently,

$$E_a(e^{-\alpha\tau_b}) = \frac{{}_1F_1\left(\frac{\alpha}{2\beta}, 1, \beta a\right)}{{}_1F_1\left(\frac{\alpha}{2\beta}, 1, \beta b\right)} \quad \text{for } 0 < a < b. \tag{3.6}$$

Lemma 3.1. *For any* $0 < \gamma \leq 1$,

$$P\left(\sup_{0 \leq a \leq \gamma W(t)} \frac{L(a, t) - L(0, t)}{L(0, t)} > w \mid W(t) > 0\right)$$

$$= \int_0^\infty e^{-v} \frac{{}_1F_1\left(\frac{1}{2\gamma}, 1, v\right)}{{}_1F_1\left(\frac{1}{2\gamma}, 1, v(1+w)\right)} \, dv. \tag{3.7}$$

Proof. By the scale change property of the Wiener process it is obvious that the left hand side of (3.7) is independent of t. Hence t can also be replaced by the exponentially distributed random variable T independent of our Wiener process. By (3.3)-(3.6) for $z > 0$ we obtain

$$P\left(\sup_{0 \leq a \leq \gamma z} \frac{L(a, T) - L(0, T)}{L(0, T)} > w \,\middle|\, W(T) = z\right)$$

$$= P\left(\sup_{0 \leq a \leq \gamma z} L(a, T) > (1+w)L(0, T) \mid W(T) = z\right)$$

$$= 1 - \int_0^\infty P_x(\tau_{(1+w)x} > \gamma z \mid W(T) = z)\beta \, e^{-\beta x} \, dx$$

$$= 1 - \int_0^\infty \beta \, e^{-\beta x} \int_{\gamma z}^\infty r_x^{(1+w)x}(u) \, du \, dx. \tag{3.8}$$

where $r_x^y(u)$ is the first passage density of the diffusion with generator (3.5), i.e.

$$r_x^y(u) = \frac{d}{du} P_x(\tau_y < u).$$

Since $W(T)$ is also exponential with parameter β, we get, from (3.6) and (3.8),

$$P\left(\sup_{0\le a\le \gamma z}\frac{L(a,t)-L(0,t)}{L(0,t)}>w\;\middle|\;W(t)>0\right)$$

$$=1-\int_0^\infty\int_0^\infty \beta\,\mathrm{e}^{-\beta x}\int_{\gamma z}^\infty r_x^{(1+w)x}(u)\,\mathrm{d}u\,\mathrm{d}x\,\beta\,\mathrm{e}^{-\beta z}\,\mathrm{d}z$$

$$=1-\beta\int_0^\infty \mathrm{e}^{-\beta x}(1-E_x(\mathrm{e}^{-(\beta/\gamma)\tau_{(1+w)x}}))\,\mathrm{d}x$$

$$=\beta\int_0^\infty \mathrm{e}^{-\beta x}\frac{{}_1F_1\left(\frac{1}{2\gamma},1,\beta x\right)}{{}_1F_1\left(\frac{1}{2\gamma},1,\beta x(1+w)\right)}\,\mathrm{d}x,$$

which gives (3.7) by substituting $v=\beta x$. □

The next lemma gives a lower bound for the probability evaluated in Lemma 3.1.

Lemma 3.2

$$P\left(\sup_{0\le a\le \gamma W(t)}\frac{L(a,t)-L(0,t)}{L(0,t)}>w\;\middle|\;W(t)>0\right)\ge\gamma K(w) \tag{3.9}$$

if $0<\gamma\le 1$, *where* $K(w)>0$ *is a constant* (*depending only on* w).

Proof. The basic tool for proving (3.9) is an asymptotic formula of Slater [14, p. 67 (4.4.1)];

$$\lim_{r\to\infty}{}_1F_1\left(r,s,\frac{x}{r}\right)=\Gamma(s)x^{1/2-s/2}I_{s-1}(2\sqrt{x}) \tag{3.10}$$

where $I_\nu(x)$ stands for the Bessel function of order ν:

$$I_\nu(z)=\sum_{m=0}^\infty\left(\frac{z}{2}\right)^{2m+\nu}\cdot\frac{1}{m!\,\Gamma(m+\nu+1)}.$$

For any $B>0$, we get the following estimation, using the monotonicity of ${}_1F_1(r,s,\cdot)$, and a simple integration formula for the confluent hypergeometric function (see Slater [14]):

$$\int_0^\infty \mathrm{e}^{-v}\frac{{}_1F_1\left(\frac{1}{2\gamma},1,v\right)}{{}_1F_1\left(\frac{1}{2\gamma},1,v(w+1)\right)}\,\mathrm{d}v\ge\int_0^B \mathrm{e}^{-v}\frac{{}_1F_1\left(\frac{1}{2\gamma},1,v\right)}{{}_1F_1\left(\frac{1}{2\gamma},1,v(w+1)\right)}\,\mathrm{d}v$$

$$\ge\frac{\int_0^B \mathrm{e}^{-v}{}_1F_1\left(\frac{1}{2\gamma},1,v\right)\mathrm{d}v}{{}_1F_1\left(\frac{1}{2\gamma},1,B(w+1)\right)}=\frac{B\,\mathrm{e}^{-B}{}_1F_1\left(1+\frac{1}{2\gamma},2,B\right)}{{}_1F_1\left(\frac{1}{2\gamma},1,B(w+1)\right)}.$$

Choose $B=2\gamma/(1+2\gamma)$; then

$$\frac{2\gamma}{1+2\gamma}\,\mathrm{e}^{-(2\gamma/(1+2\gamma))}\frac{{}_1F_1\left(1+\frac{1}{2\gamma},2,\frac{2\gamma}{1+2\gamma}\right)}{{}_1F_1\left(\frac{1}{2\gamma},1,\frac{2\gamma}{1+2\gamma}(w+1)\right)}$$

$$\geqslant \gamma\,\mathrm{e}^{-1}\frac{{}_1F_1\left(\frac{2\gamma+1}{2\gamma},2,\frac{2\gamma}{2\gamma+1}\right)}{{}_1F_1\left(\frac{1}{2\gamma},1,2\gamma(w+1)\right)}. \tag{3.11}$$

Now apply (3.10) with $x=1$ for the numerator and with $x=w+1$ for the denominator to get

$$\frac{\gamma}{\mathrm{e}}\frac{F\left(\frac{2\gamma+1}{2\gamma},2,\frac{2\gamma}{2\gamma+1}\right)}{F\left(\frac{1}{2\gamma},1,2\gamma(w+1)\right)}\sim\frac{\gamma}{\mathrm{e}}\frac{I_1(2)}{I_0(2\sqrt{w+1})}\quad\text{as }\gamma\to 0,$$

proving (3.9) □

We need for our proof the following result (Csáki and Földes [5, Lemma 4.3]). Let $t_k>0$, for $k=1,2,\dots$ and define the following stopping times:

$$\begin{aligned}
&\eta_0=0,\\
&\eta_1=\inf\{t\colon t>t_1,\ W(t)=0\},\\
&\quad\vdots\\
&\eta_k=\inf\{t\colon t>\eta_{k-1}+t_k,\ W(t)=0\}.
\end{aligned} \tag{3.12}$$

Put, furthermore,

$$\alpha_k=\frac{1}{t_k}(\eta_k-\eta_{k-1}),\quad k=1,2,\dots.$$

Theorem D. $\{\alpha_k\}_{k=1}^{\infty}$ *is a sequence of i.i.d random variables, such that, for any* $u>0$,

$$P\left(\sum_{i=1}^{k}\alpha_i>u\right)\leqslant\frac{3k}{\sqrt{u}},\quad k=1,2,\dots. \tag{3.13}$$

Turning to the Proof of Theorem 2(ii); Let

$$\begin{aligned}
&t_1=1,\qquad t_k=\mathrm{e}^{\delta k\log k},\quad k=2,3,\dots,\\
&T_k=\eta_{k-1}+t_k,\quad k=1,2,\dots,
\end{aligned} \tag{3.14}$$

where δ is a suitably chosen constant and η_k is defined by (3.12).

Define the variables

$$L_k^* = \begin{cases} \sup\limits_{0 \le a \le cW(T_k)/\log T_k} L(a, T_k) & \text{if } W(T_k) > 0, \\ \sup\limits_{cW(T_k)/\log T_k \le a \le 0} L(a, T_k) & \text{if } W(T_k) < 0, \end{cases} \tag{3.15}$$

and let

$$A_k = \{L_k^* > L(0, T_k)(1+w)\}.$$

Our aim is to show that, for any $w > 0$,

$$P(A_k \text{ i.o.}) = 1 \tag{3.16}$$

which clearly implies our statement, since

$$\frac{L_k^*}{L(0, T_k)} - 1 \le \sup_{cm(T_k)/\log T_k \le a \le cM(T_k)/\log T_k} \left| \frac{L(a, T_k)}{L(0, T_k)} - 1 \right|.$$

To prove (3.16) we are going to apply the divergent part of Borel-Cantelli lemma. However the events A_k are not independent. Therefore we define independent events B_k, by

$$B_k = \{\hat{L}_k^* > 2(L(0, T_k) - L(0, \eta_{k-1}))(1+w)\},$$

where

$$\hat{L}_k^* = \begin{cases} \sup\limits_{0 \le a \le (c/2)(W(T_k) - W(\eta_{k-1}))/\log t_k} (L(a, T_k) - L(a, \eta_{k-1})) & \text{if } W(T_k) > 0, \\ \sup\limits_{(c/2)(W(T_k) - W(\eta_{k-1}))/\log t_k \le a \le 0} (L(a, T_k) - L(a, \eta_{k-1})) & \text{if } W(T_k) < 0, \end{cases} \tag{3.17}$$

and t_k, η_k, T_k are defined by (3.12) and (3.14).

Since $T_k - \eta_{k-1} = t_k$, from (3.7) and (3.9) we obtain, by the symmetry of the Wiener process,

$$P(B_k) \ge \frac{c^*}{\log t_k} = \frac{c^*}{\delta k \log k}.$$

Hence $\sum_k P(B_k)$ diverges, and since B_k are independent, we have by Borel-Cantelli lemma that

$$P(B_k \text{ i.o.}) = 1. \tag{3.18}$$

Next we show that (3.18) implies (3.16). To see this, observe that

$$\begin{aligned} T_k = t_k + \eta_{k-1} &= t_k + \sum_{j=1}^{k-1} (\eta_j - \eta_{j-1}) = t_k + \sum_{j=1}^{k-1} t_j \alpha_j \\ &\le t_k + t_{k-1} \sum_{j=1}^{k-1} \alpha_j, \end{aligned}$$

and, by Theorem C,

$$P\left(\sum_{j=1}^{k-1} \alpha_j > (k-1)^6\right) \leqslant \frac{3}{(k-1)^2},$$

so that

$$t_k \leqslant T_k \leqslant t_k + t_{k-1} k^6 \quad \text{a.s.} \tag{3.19}$$

if k is large enough. Therefore

$$t_k \leqslant T_k \leqslant t_k(1 + k^{6-\delta}) \quad \text{a.s.} \tag{3.20}$$

By choosing $\delta > 6$, we have

$$\log T_k \sim \log t_k \quad \text{a.s.}\ (k \to \infty), \tag{3.21}$$

and consequently

$$L_k^* \geqslant \hat{L}_k^* \quad \text{a.s.} \tag{3.22}$$

if k is large enough, since $W(\eta_{k-1}) = 0$. What remains to show is that

$$L(0, T_k) < 2(L(0, T_k) - L(0, \eta_{k-1})) \quad \text{a.s.} \tag{3.23}$$

for large k, i.e.

$$L(0, T_{k-1}) = L(0, \eta_{k-1}) < \tfrac{1}{2} L(0, T_k) \quad \text{a.s.}$$

By Kesten's law of the iterated logarithm [11] we have, using (3.20),

$$\begin{aligned} L(0, T_{k-1}) &\leqslant L(0, 2t_{k-1}) \leqslant C_1 (t_{k-1} \log\log t_{k-1})^{1/2} \\ &\leqslant c_1\, \mathrm{e}^{(\delta/2)(k-1)\log k} \log(\delta k \log k) \\ &\leqslant c_1 t_k^{1/2} k^{-\delta/2} \log(\delta k \log k). \end{aligned}$$

On the other hand, by Chung and Hunt [4],

$$L(0, T_k) \geqslant L(0, t_k) \geqslant t_k^{1/2} (\log t_k)^{-2} = t_k^{1/2} (\delta k \log k)^{-2},$$

and hence we have (3.23) by the choice of $\delta > 6$. Now (3.22), (3.23) and (3.18) imply (3.16), so the proof of Theorem 2 (ii) is complete. □

Acknowledgement

The authors are indebted to the referee for valuable remarks.

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